Game Theory Exam 2018 Cheat Sheet

Non- Cooperative Game Theory

Players: $N = 1, 2, .., n$
Actions/strategies: Each player chooses s_i from his own finite strategy set: S_i for each $i \in N$ resulting in a tuple that describes strategy combination: $s = (s_1, ..., s_n) \in (S_i)_{i \in N}$
Payoff outcome: $u_i = u_i(s)$ for some chosen strategy
best-response: Player i's best-response to the strategies s_{-i} played by all others is the strategy $s_i^* \in S_i$ such that

$$u_i(s_i^*, s_i) \geq u_i(s_i^{'}, s_{-i}) \forall s_i^{'} \in S_i and s_i^{'} \neq s_i^*$$

Pure - strategy (Nash Equilibrium): All strategies are mutual best responses:

$$u_i(s_i^*, s_i) \geq u_i(s_i^{'}, s_{-i}) \forall s_i^{'} \in S_i and s_i^{'} \neq s_i^*$$

Cooperative game

Population of players: $N = 1, 2, ..., n$
Colations: $C \subset N$ form in the population and **become players** results in a coalition structure: $\rho = \{C_1, C_2, ..., C_k\}$
Payoffs: $\Phi = \{\Phi_1, ..., \Phi_n\}$ and we need a sharing rule for for the individual player resulting in: $\Phi_i = \Phi(\rho, "sharing\, rule")$
characteristic function form (CFG): The Game is defined by 2-tuple $G(v, N)$ where **Characteristic function:** $v : 2^N \to R$ where 2^N are all possible coalitions.
transfer of utils: occures when we share the value of the characteristic function among the participants of the coalition. and **feasibility** is then when: $\sum_{i \in C} \Phi_i \leq v(C)$
Superadditivity: If two coalitions C, S are disjoint then $v(C) + v(S) \leq v(C \cup S)$
The Core of a superadditive G(v, N) consists of all outcomes where the grand coalition forms **and** payoff allocations $\Phi = (\Phi_1, ..., \Phi_n)$ are:

1. Pareto efficient: $\sum_{i \in N} \Phi_i = v(N)$ so whole value is out
2. Unblockable: $\forall C \subset N, \sum_{i \in C} \Phi_i \geq v(C)$ so payout for each individual is bigger then if it would act alone(individual rational) or in a sub-coalition(coalitional rational) of N.

nonempty core: if and only if the game is balanced.
balancedness:

1. *Balancing weight:* attached to each Coalition C: $\alpha(C) \in [0, 1]$
2. *Balanced family:* A set of balancing weights is balanced family if $\sum_{i \in C} \alpha(C_i) = 1$
3. *Balancedness in superadditivity:* requires that for all balanced families: $v(N) \geq \sum_{C \in 2^N} \alpha(C) v(C)$

Shapley value

Pays each player average marginal contributions. **Marginal contributions:** For any S: $i \in S$, think of marginal contributions as : $MC_i(S) = v(S) - v(S \setminus i)$
Given some $G(v, N)$, an acceptable allocation/value $x^*(v)$ should satisfy:

1. **Efficiency:** $\sum_{i \in N} x_i^*(v) = v(N)$
2. **Symmetry:** if for any two players i and j, $v(S \cup i) = v(S \cup j)$ so player i,j same influence on value v(S). then $x_i^*(v) = x_j^*(v)$
3. **Dummy player** if for any i, $v(S \cup i) = v(S)$ for all S then $x_i^*(v) = 0$
4. **Additivity** if u,v are two characteristic functions then $x^*(v + u) = x^*(v) + x^*(u)$

Shapley function:
$\Phi_i(v) = \sum_{S \in N, i \in S} = \frac{(|S|-1)!(n-|S|)!}{n!}(v(S) - v(S \setminus i))$. It pays *average marginal contributions.*
Non transferable-utility cooperative game: Game: $G(v, N)$ **outcome:** partition $\rho = C_1, C_2, ..., C_k$ implies directly a payoff allocation. e.g. only coalitions of pairs. **Deferred acceptance:** For any marriage problem, one can make all matchings stable using the deferred acceptance algorithm.
1 Initialize) all $m_i \in M$ and all $w_i \in W$ are single.
2 Engage) Each single man $m \in M$ proposes to his *preferred* woman w to whom he has not yet proposed. a) If w is single, she will become engaged with her *preferred* proposer. b) Else w is already engaged with m': if w prefers proposer m over m' she becomes engaged with m and m' becomes single. If not (m', w) remain engaged. c) All proposers wo do not become engaged remain single. **3 Repeat)** If there exists a single man after Engage repeat Engage. Else Terminate **4 Terminate)** Marry all engagements

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Preferences and utility

A binary relation $\succeq$ (weakly prefers), $\succ$ (prefers), $\sim$ (indifferent) on a set X is a non-empty subset $P \subset X \times X$. We write $x \succeq y$ iff $(x, y) \in P$
Assumptions on preferences:

1. Completeness: $\forall x, y \in X : x \succeq yory \succeq xorboth$ so we have some preference for any element to any other in the set.
2. Transitivity: $\forall x, y, z \in X$: if $x \succeq y$ and $y \succeq z$ then $x \succeq z$
3. Continuity: $W(x) = y \in X : x \succeq y, B(X) = y \in X : y \succeq x$ so we have once all below x and once all above x then continuity tells us that we do not have some kind of big gap or between these. $\forall x \in X : B(x)$ and $W(x)$ are closed sets.(including their boundary points)
4. Independence of irrelevant alternatives: $\forall x, y, z \in X$: $x \succ y \Rightarrow (1 - \lambda)x + \lambda z \succ (1 - \lambda)y + \lambda z = x + z \succ y + z$

A utility function for a binary relation $\succeq$ on a set X is a function $u : X \to \mathbb{R}$ such that

$$u(x) \geq u(y) \Leftrightarrow x \succeq y$$

so give the preference an actual value and still preserving the preference. There exists such a utility function for each complete, transitive, positively measurable and continuous preference on any closed or countable set.
Ordinal utility function: difference between u(x) and u(y) is meaningless. Only $u(x) \geq u(y)$ is meaningful.
Cardinal utility function: A utility function where differences between $u(x)$ and $u(y)$ are meaningful as they reflect the intensity of preferences. (invariant to positive affine transformations)
Utils: An even stronger statement would be that there is a fundamental measure of utility. say one "util". It is not invariant to any transformation.
Lottery Let X be a set of outcomes then a lottery on X means nothing but a probability distribution on X. The set of all lotteries on X is usually denoted by $\Delta(X)$. E.g. $X = (x_1, ..., x_K)$ then a lottery is represented by $(p_1, ..., p_K)$ and they should sum to one.
Decision problem under risk: Is then when the decision maker has to choose a lottery from a Set of available lotteries: $C \subseteq \Delta(X)$
St. Petersburg Paradox: A rational decider would prefer lotteries with higher expected payoff. E[l] ¿ E[l'] but this leads to a paradox when using infinity expected values.
Expected utility maximization: Was introduced to solve St. Petersburg Paradox. So instead of weighting lotteries directly on their payoff we weight them on their utility function.
Utility function on lotteries: A preference relation $\succeq$ on $\Delta(X)$ is sait to be representable by a utility function U whenever for every lotteries $p := (p_1, ..., p_k)$ and $p' := (p'_1, ..., p'_k)$, $p \succeq p'$ only when $U(p) \geq U(p')$
Bernouilli function is the utility function over the outcomes of the lottery. So $X = (x_1, ..., x_K)$ then bernouilli function is: $u : X \to \mathbb{R}_+$ by considering all the axioms that hold for utility functions.
Expected utility function: Is a utility function on the set of $\Delta(X)$ of utilities.
Bernouilli function / von Neumann morgenstern utility function: If $\succeq$ is a binary relation on X representing the agent's preferences over lotteries over T. If there is a function $v : T \to \mathbb{R}$ such that

$$x \succeq y \Leftrightarrow \sum_{k=1}^{m} x_k v(\tau_k) \geq \sum_{k=1}^{m} y_k v(\tau_k)$$

then

$$u(x) = \sum_{k=1}^{m} x_k v(\tau_k)$$

where v is called a *Bernouilli function*, and where x_i are the probabilities of event τ_i happening
Existence of Neumann-Morgenstern utility function: Let $\succeq$ be a complete, transitive and continuous preference

relation on $X = \nabla(T)$ for any finite set T. Then $\succeq$ admits a utility function u of the expected-utility form iff $\succeq$ meets the axiom of independence of irrelevant alternatives.
Sure thing principle (Savage): A decision maker who would take a certain Action A if he knew that event B happens should also take Action A if he new that B not happens and also if he knew nothing about B. (This is equivalent to independence of irrelevant alternatives)
Risk neutral: An agent is risk-neutral iff he is indifferent between accepting and rejecting all fair gambles that is for all α, τ_1, τ_2 :

$$\mathbb{E}[u(lottery)] = \alpha \cdot v(\tau_1) + (1-\alpha) \cdot v(\tau_2) = u(\alpha\tau_1 + (1-\alpha)\tau_2)$$

Risk averse: An agent is risk averse iff he rejects all fair gambles for all α, τ_1, τ_2 :

$$\mathbb{E}[u(lottery)] = \alpha \cdot v(\tau_1) + (1-\alpha) \cdot v(\tau_2) < u(\alpha\tau_1 + (1-\alpha)\tau_2)$$

Since $g(\lambda\alpha + (1-\lambda)\beta) > \lambda g(\alpha) + (1-\lambda)g(\beta)$ is the def. of concavity to be risk averse the utility function has to be strictly concave.
Risk seeking: An agent is risk seeking iff he strictly prefers all fair gambles for all α, τ_1, τ_2 :

$$\mathbb{E}[u(lottery)] = \alpha \cdot v(\tau_1) + (1-\alpha) \cdot v(\tau_2) > u(\alpha\tau_1 + (1-\alpha)\tau_2)$$

Since $g(\lambda\alpha + (1-\lambda)\beta) < \lambda g(\alpha) + (1-\lambda)g(\beta)$ is the def. of convexity to be risk seeking the utility function has to be strictly convex.

Normal form games

Normal form:

1. Players: $N = 1, ..., n$
2. Strategies: For every player i, a finite set of strategies, S_i with typical strategy $s_i \in S_i$.
3. Payoffs: A function $u_i : (s_1, ..., s_n) \to \mathbb{R}$ mapping strategy profiles to a payoff for each player i. $u : S \to \mathbb{R}^n$

Normal form triplet: $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$
Strategy profile: $s = (s_1, ..., s_n)$ is called a strategy profile. Is a collection of strategies, one for each player. If s is played, player i receives $u_i(s)$
Opponents strategies: Write s_{-i} for all strategies except for the one of player i. So a strategy profile may be written as $s = (s_i, s_{-i})$ **Dominance:**

1. Strict Dominance: s_i strictly dominates s_i‘ if $u_i(s_i, s_{-i}) > u_i(s_i‘, s_{-i}) \forall s_{-i}$
2. Weak Dominance: s_i strictly dominates s_i‘ if $u_i(s_i, s_{-i}) \geq u_i(s_i‘, s_{-i}) \forall s_{-i}$
3. Dominated: A strategy s_i‘ is strictly dominated if there is an s_i that strictly dominates it.
4. A strategy s_i is strictly dominant if it strictly dominates all $s_i‘ \neq s_i$

So obvs. we do not play a dominated strategy no matter what others are doing.
Dominant-Strategy Equilibrium: The strategy profile s^* is a dominant-strategy equilibrium if for every player i, $u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$ for all strategy profiles $s = (s_i, s_{-i})$
Nash Equilibrium: is a strategy profile s^* such that for every player i,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \forall s_i$$

So no player has any regrets hi could not have done better when all other played like they have.
Best reply function:
$B_i(s_{-i}) = \{s_i | u_i(s_i, s_{-i}) \geq u_i(s_i‘, s_{-i}) \forall s_i^‘$ given the actions from our opponents chose our best action. and with the best function the **Nash equilibrium** gets: s^* is a **Nash equilibrium** iff $s_i^* \in B_i(s_{-i}^*) \forall i$
Mixed strategy: A mixed strategy σ_1 for a player i is any probability distribution over his or her set S_i of pure strategies. The set of mixed strategies is:

$$\delta(S_i) = \{x_i \in \mathbb{R}_+^{|S_i|} : \sum_{h \in S_i} x_{ih} = 1$$

Mixed extension The mixed extension of a game G has players, strategies and payoffs: $\Gamma = (N, \{S_i\}_{i\in N}, \{U_i\}_{i\in N})$ where

1. Strategies are **probability distributions** in the the set $\delta(S_i)$ meaning we do not choose a strategy deterministically but we choose a strategy according to the distribution σ_i
2. U_i is player $i‘s$ expected utility function assigning a real number to every strategy profile $\gamma = (\gamma_1, ..., \gamma_n)$. $U_i(\sigma) = \sum_s u_i(s) \prod_{j\in N} \sigma_j(s_j)$

All the things like best response and nash equilibrium hold also for mixed strategies!
How to find mixed Nash equilibria:

1. Find all pure strategy Nash equilibria.
2. Check wether there is an equilibrium in which row mixes between several of her strategies:
   (a) Identify candidates:
      i. if there is such an equilibrium then each of these strategies must yield the same expected payoff given column's equilibrium strategy.
      ii. Write down these payofss ans olve for column's equilibrium mix.
      iii. Reverse: Look at the strategies that column is mixing on and solve for rows equilibrium mix.
   (b) Check candidates:
      i. The equilibrium mix we found must indeed involve the strategies for row we started with.
      ii. All probabilities we found must indeed be probabilities (between 0 and 1)
      iii. Neither player has a positive deviation

Nash Theorem: Every finite game has at least one "Nash" Equilibrium in mixed strategies!

Games / Extensive form

Trembling hand / perfect equilibrium: Take the strategy that is played by nash equilibrium. E.g. (a1, b1) then check if $E[u(a1)] = u(a1, b1)(1-\epsilon) + u(a1, b2)\epsilon$ where b2 is played with some error probability ϵ. And despite the error this must be greater then as if player 1 would play a2 instead: $E[u(a2)] = u(a2, b1)(1-\epsilon) + u(a2, b2)\epsilon$ if $E[u(a1)] > E[u(a2)]$ the nash equilb. is perfect equilibrium.
Extensive form game: Players, Basic structure is a game tree with nodes $a \in A, a_0$ is root of tree. And nodes can bee Decision nodes where a player makes a decision or a Chance node where nature plays according to some probability distribution. (If 2 nodes are connected we have no information from above and we have to decide based on nash equilibrium.)
Subgames if a node has been reached we have full information at its root node and we can decide as if we're in isolation.
Strategy set in subgame: A strategy for a player is over the whole game so if we have 2 subgames with possibility a,b then we have (a,a), (a, b), (b, a), (b, b) where we simultaneously describe as we would play both subgames despite the fact that we play only 1. (important if we have the same information in 2 subgames (shown via connection in between the subnodes) then we have only 2 strategies(a,b).

Evolutionary Game Theory

Symmetric two-player games: $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}$.

1. Players: $N = \{1, 2\}$
2. Strategies: $S_1 = S_2 = S$ with typical strategy $s \in S$
3. Payoffs: A function $u_i : (h, k) \to \mathbb{R}$ mapping strategy profiles to a payoff for each player i such that for all $h, k \in S$: $u_2(h, k) = u_1(k, h)$. So it does not really matter what action is chosen the payoff stays the same for both players (in good or bad).

Symmetric Nash Equilibrium: is a strategy profile σ^* such that for every player i,

$$u_i(\sigma^*, \sigma^*) \geq u_i(\sigma, \sigma^*) \forall \sigma$$

, "if no player has an incentive to deviate from their part in a particular stategy profile, then it is Nash Equilibrium.
In a symmetric normal form game there always exists a symmetric Nash Equilibrium. (Not all Nash Equilibria of a symmetric game need to be symmetric.
Evolutionary stable strategy(ESS): A mixed strategy $\sigma \in \delta(S)$ is an *evolutionary stable strategy* (ESS) if for every strategy $\tau \neq \sigma$ there exists $\epsilon(\tau) \in (0, 1)$ s.t. $\forall \epsilon \in (0, \epsilon(\tau))$:

$$U(\sigma, \epsilon\tau + (1-\epsilon)\sigma) > U(\tau, \epsilon\tau + (1-\epsilon)\sigma)$$

A **mixed strategy** $\sigma in \delta(S)$ is an evolutionary stable strategy if:

$$U(\tau,\sigma) \leq U(\sigma,\sigma) \forall \tau$$
$$U(\tau,\sigma) = U(\sigma,\sigma) \Rightarrow U(\tau,\tau) < U(\sigma,\tau) \forall \tau \neq \sigma$$

Experimential Game Theory

perfect rationality

1. Common knowledge about the structure of the game and the payoffs
2. Common beliefs: players have beliefs about each others behaviour these believes are correct
3. Optimization: individual behavior is governed by optimization / maximization in terms of expected utilities.

Pure self-interest:

1. narrow self interest: agent cares about own material payoff only
2. no concern for other players payoff
3. no consideration of the effects of his actions on upholding higher order norms or similar
4. decisions are not subject so social influence

More realisticly: Players follow norms / social influences, care about others. Also they know little about how their choices will affect others and know little about the overall game.

Ultimatum Game: One side proposes moves first. Makes a proposal as to how to split a cake. Other side eather accepts and both get their share or rejects and both get zero. Nash equilibria: Any proposal made responder accepts. Subgame perfection: proposer takes all, accept nevertheless.

Public good game: Choose how much you want to contribute between 0 and 20. What you don't contribute is autom. yours. The whole contribution gets multiplied by 3 and equal shares get paid back to all players. Nash equilibrium : Universal non-contribution.
